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A ZERO-ENTROPY MIXING TRANSFORMATION WHOSE PRODUCT WITH ITSELF IS LOOSELY BERNOULLI

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ABSTRACT
A zero-entropy mixing transformation T is constructed such that $T \times T$ is loosely Bernoulli (LB). Previously known examples were not mixing. The construction is then generalized to yield a zero-entropy mixing transformation T such that the n-fold product $T \times \cdots \times T$ is LB for each positive integer n. Furthermore, a flow with the same properties is obtained.

§1. Introduction

A few years ago, in connection with the problem of classifying measure-preserving flows up to a time change on the orbits, the loosely Bernoulli (LB) property was introduced by J. Feldman [2] and, independently, by A. Katok [4] (in the case of zero entropy). (See also E. Sataev [13].) The LB property is obtained by replacing the Hamming metric, $\bar{d}$, on sequences of symbols in D. Ornstein's very weak Bernoulli (VWB) property by a coarser metric, now called $\bar{f}$.

If $\alpha = (\alpha_0, \cdots, \alpha_{N-1})$, $\beta = (\beta_0, \cdots, \beta_{N-1})$ are sequences of symbols, then

$$\bar{f}_N(\alpha, \beta) = 1 - \frac{1}{N} \max\{k : \text{there exist sequences}$$

$$(i_1, \cdots, i_k) \quad \text{and} \quad (j_1, \cdots, j_k) \text{ such that}$$

$$0 \leqq i_1 < \cdots < i_k \leqq N-1, \quad 0 \leqq j_1 < \cdots < j_k \leqq N-1$$

$$\text{with } \alpha_{i_m} = \beta_{j_m}, 1 \leqq m \leqq k\},$$

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i.e., we take the best possible match allowing some "stretching" of the sequences. For a zero-entropy process $(T, \mathcal{P})$, the LB property reduces to the following: for every $\varepsilon > 0$ there exists a positive integer N and a set A of measure greater than $1 - \varepsilon$ such that $\bar{f}_N(\alpha, \beta) < \varepsilon$ whenever $\alpha = (\alpha_0, \cdots, \alpha_{N-1})$, $\beta = (\beta_0, \cdots, \beta_{N-1})$ are the T-$\mathcal{P}$-N-names of points in A. (Here symbols are identified with elements of $\mathcal{P}$, and a point x has T-$\mathcal{P}$-N-name $\alpha = (\alpha_0, \cdots, \alpha_{N-1})$ if $T^i x \in P_{\alpha_i} \in \mathcal{P}$ for $0 \leqq i \leqq N-1$.) A transformation T is said to be LB if $(T, \mathcal{P})$ is an LB process for every finite partition $\mathcal{P}$.

The LB property is preserved under Kakutani equivalence [2], [13], and an LB transformation is Kakutani equivalent to a Bernoulli shift if it has positive entropy and to an ergodic group rotation if it has entropy zero [5], [13], [16].

The class of LB transformations behaves well under most functorial operations, but Ornstein showed that $T \times T$ need not be LB even if T is LB (see D. Rudolph [12]). The difficulty in getting $T \times T$ to be LB is that pairs of sequences of symbols must be matched, and once we decide how to make an $\bar{f}$ match for the first sequences in the pairs (i.e., how to choose $i_1, \cdots, i_k$ and $j_1, \cdots, j_k$ in the definition above), then the "stretching" in the second sequences in the pairs is forced. The problem of getting LB Cartesian products of transformations was studied by L. Swanson [14], and M. Ratner showed that if T is any transformation from the horocycle flow, then T is LB [9], but $T \times T$ is not LB [10]. Then A. Katok pointed out, in a conversation, a simple example of a zero-entropy weakly mixing T with $T \times T$ LB. However, his transformation is not mixing, and it seems more difficult to get such an example to be mixing. In this paper, we combine Katok's construction with a construction introduced by A. Rothstein [11] to obtain a zero-entropy transformation whose product with itself is LB and which, in addition, satisfies the "Veršik property." This property is a weakening of the VWB property and it implies mixing of all orders. Thus, we obtain a zero-entropy T which is mixing of all orders and such that $T \times T$ is LB. L. Swanson [15] has extended Katok's method to obtain a zero-entropy T such that *each* n-fold product $T \times \overset{n}{\cdots} \times T$ is LB. Her example, like Katok's, is not mixing. Here we obtain such an example which is mixing of all orders. Finally, we build a flow analogue of this example.

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§2. Katok's example

This transformation is built by a cutting and stacking argument that is a modification of Chacon's method of obtaining a weakly mixing transformation that is not mixing [1]. We will describe Katok's example, S, inductively, in terms of the parameters $g_1, g_2, \cdots$. Let the first tower simply have one level, which we take to be an interval. Label the points in this interval 1. To obtain the $(n+1)$th tower from the nth tower, first add an interval over the right half of the nth tower and label the points in the added interval 0. Then divide this modified nth tower into $2g_n$ subcolumns of equal width. The $(n+1)$th tower is the result of stacking the $(i+1)$th subcolumn above the ith subcolumn for $1 \leqq i \leqq 2g_n - 1$. For $n \geqq 1$, S maps each point in the nth tower which is not in the top level to the point directly above it in the next level. (See Fig. 1.) Let Y_n be the union of the intervals in the nth tower and let $Y = \bigcup_1^\infty Y_n$. Then Y has finite Lebesgue measure, and S is defined almost everywhere on Y. Since S has rank 1, it has entropy zero. By the same argument that is used for Chacon's example, S is weakly mixing, but not mixing. Let $\mathcal{Q}$ be the partition of Y into two sets, according to which of the labels 0, 1 a point has. Then $\mathcal{Q}$ is a generator (for the Lebesgue measurable sets) under S, and if $g_1, g_2, \cdots$ grow sufficiently rapidly, then $(S, \mathcal{Q})$ is LB. This last assertion follows easily from an equivalent formulation of the zero-entropy LB property given in proposition 1 of Katok and Sataev [6].

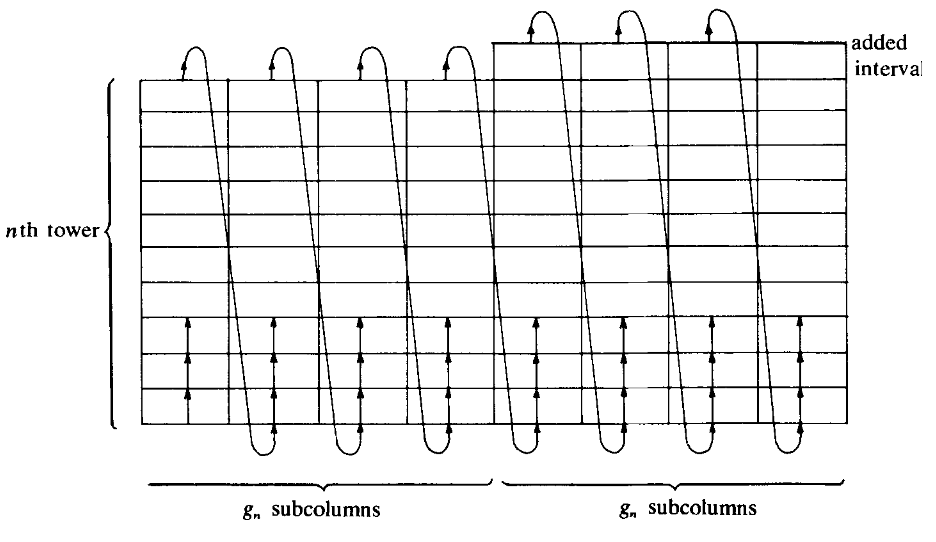


Fig. 1

§3. The Veršik property

We say that a process $(U, \mathcal{R})$ is a Veršik process (or, briefly, a V process) if, for every $\varepsilon > 0$, there is an N such that if $L, M > N$ then

$$\bar{e}\left(\bigvee_{-L}^{M} U^{-i}\mathcal{R}, \bigvee_{-L}^{0} U^{-i}\mathcal{R} \times \bigvee_{1}^{M} U^{-i}\mathcal{R}\right) < \varepsilon.$$

The $\bar{e}$ distance between the ordered partitions $\bigvee_{-L}^{M} U^{-i}\mathcal{R}$ and $\bigvee_{-L}^{0} U^{-i}\mathcal{R} \times \bigvee_{1}^{M} U^{-i}\mathcal{R}$ is defined analogously to the $\bar{d}$ distance, with the $\bar{d}$ distance on names replaced by

$$\bar{e}((\alpha_{-L}, \cdots, \alpha_0, \cdots, \alpha_M), (\beta_{-L}, \cdots, \beta_0, \cdots, \beta_M))$$
$$= \max\left[\bar{d}((\alpha_{-L}, \cdots, \alpha_0), (\beta_{-L}, \cdots, \beta_0)), \bar{d}((\alpha_1, \cdots, \alpha_M), (\beta_1, \cdots, \beta_M))\right].$$

A simple argument shows that this property is equivalent to the following: for every $\varepsilon > 0$, there is an N such that if $L > N$ then

$$\bar{d}\left(\bigvee_{-L}^{L} U^{-i}\mathcal{R}, \bigvee_{-L}^{0} U^{-i}\mathcal{R} \times \bigvee_{1}^{L} U^{-i}\mathcal{R}\right) < \varepsilon.$$

A transformation U will be called a Veršik transformation (or, briefly, a V transformation) if $(U, \mathcal{R})$ is a V process for each finite partition $\mathcal{R}$. It is easy to check that products, factors, and $\bar{d}$-limits of V processes are V; also, powers and roots of V transformations are again V transformations. It is also easy to see that a VWB process is a V process. Veršik conjectured (in a private communication to D. Ornstein) that the converse might also be true, but A. Rothstein, in his Ph.D. thesis written with Ornstein, obtained counterexamples to this conjecture [11]. Rothstein found both LB and non-LB examples of V transformations of zero entropy, and he also has an example of a V transformation which is K but not Bernoulli.

The V property generalizes to flows in a very natural way. We say that a flow $\{\phi_t : t \in \mathbf{R}\}$ has property V if for every finite partition $\mathcal{P}$,

$$\lim_{n\to\infty} \bar{d}\left(\bigvee_{t\in[-n,n]} \phi_{-t}\mathcal{P}, \bigvee_{t\in[-n,0]} \phi'_{-t}\mathcal{P} \times \bigvee_{t\in(0,n]} \phi''_{-t}\mathcal{P}\right) = 0,$$

where ϕ'_t, ϕ''_t are defined on $X \times X$ by

$$\phi'_t(x, y) = (\phi_t(x), y) \quad \text{and} \quad \phi''_t(x, y) = (x, \phi_t(y)).$$

The following are equivalent:

(i) The flow $\{\phi_t : t \in \mathbf{R}\}$ has property V;

(ii) ϕ_t is a V transformation for each $t \neq 0$;

(iii) ϕ_{t_0} is a V transformation for some t_0.

The equivalence of (ii) and (iii) was already observed by Rothstein. These arguments are based on standard techniques of passing from continuous names to discrete names and back (see, e.g., D. Lind [7]).

Rothstein's examples can all be generalized to flows. Here we construct an example of a flow $\{\phi_t : t \in \mathbf{R}\}$ which is V (and hence mixing of all orders) such that each product flow $\{\phi_t \times \cdots \times \phi_t : t \in \mathbf{R}\}$ is LB.

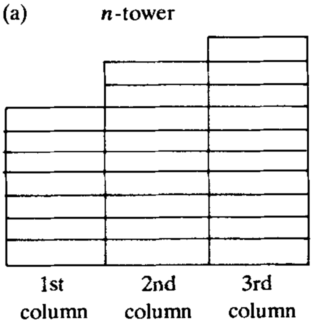
§4. Construction of a zero-entropy V transformation T such that $T \times T$ is LB

Let $\alpha_1, \alpha_2, \cdots$ be a sequence of rational numbers in $(0, \frac{1}{4})$ such that $\lim_{n\to\infty} \alpha_n = 0$ and $\Sigma_1^\infty \alpha_n^4 = \infty$. Let $\delta_1, \delta_2, \cdots$ be a strictly decreasing sequence of numbers in $(0, \frac{1}{10})$ such that $\Sigma_1^\infty \sqrt{\delta_n} < \infty$. We will describe our transformation, T, in terms of $\alpha_1, \alpha_2, \cdots, \delta_1, \delta_2, \cdots$, and the positive integer parameters $h_0, h'_1, h_1, h'_2, h_2, \cdots$, on which some conditions will later be imposed.

We obtain T from a "block construction" as follows. A 0-block consists either of the pair of symbols (1,2) or a single symbol chosen from $\{3,4,\cdots,h_0\}$, where $h_0 \geqq 3$. For $n \geqq 0$, we construct $(n+1)'$-blocks from n-blocks, and $(n+1)$-blocks from $(n+1)'$-blocks. Here and throughout this paper we will take all n-blocks [n'-blocks] to be equally probable. By this we mean that the conditional probability of being in a particular n-block [n'-block] given that we are at the beginning of some n-block [n'-block] is independent of the particular n-block [n'-block] selected. (Or, equivalently, the columns in the corresponding n-tower [n'-tower], to be constructed below, will all have the same width.) Let r_n [r'_n] be the number of n-blocks [n'-blocks]. We require that h'_{n+1} be chosen so that $\alpha_{n+1}h'_{n+1}$ is a multiple of r_n, for each $n \geqq 0$. To build the $(n+1)'$-blocks, first order the n-blocks in some manner and let $\alpha_{n+1}h'_{n+1}$ n-blocks go in cyclically, i.e. the n-blocks are listed in order $\alpha_{n+1}h'_{n+1}/r_n$ times; then let $\alpha_{n+1}h'_{n+1}$ n-blocks go in cyclically with a 0 added at the end of each cycle of n-blocks; finally, let $(1-2\alpha_{n+1})h'_{n+1}$ n-blocks go in independently. The $(n+1)$-blocks are constructed by concatenating h_{n+1} $(n+1)'$-blocks independently.

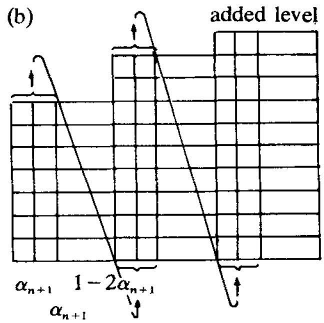
This construction may be realized by cutting and stacking intervals. We will indicate briefly how this goes; for details, see [3]. Begin by partitioning the unit interval into $h_0 - 1$ intervals of equal size, and label them $1,3,4,5,\cdots,h_0$. Above the interval labeled 1, add another interval, and label it 2. The result is what we

(a)



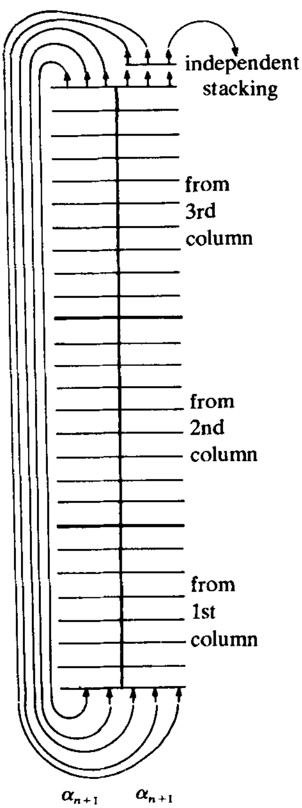
(For simplicity we are assuming $r_n = 3$, i.e. the n-tower has three columns.) Order the columns in some way.

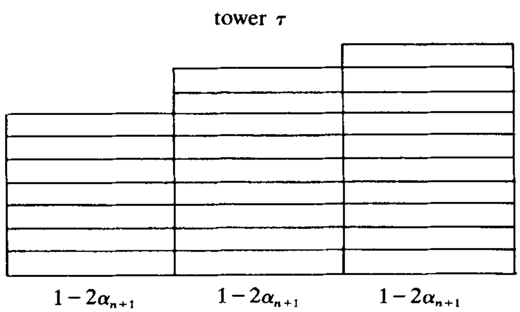
(b)



Divide each column into three subcolumns with proportions α_{n+1}, α_{n+1}, $1-2\alpha_{n+1}$, and add a level above the second subcolumn of the last column.

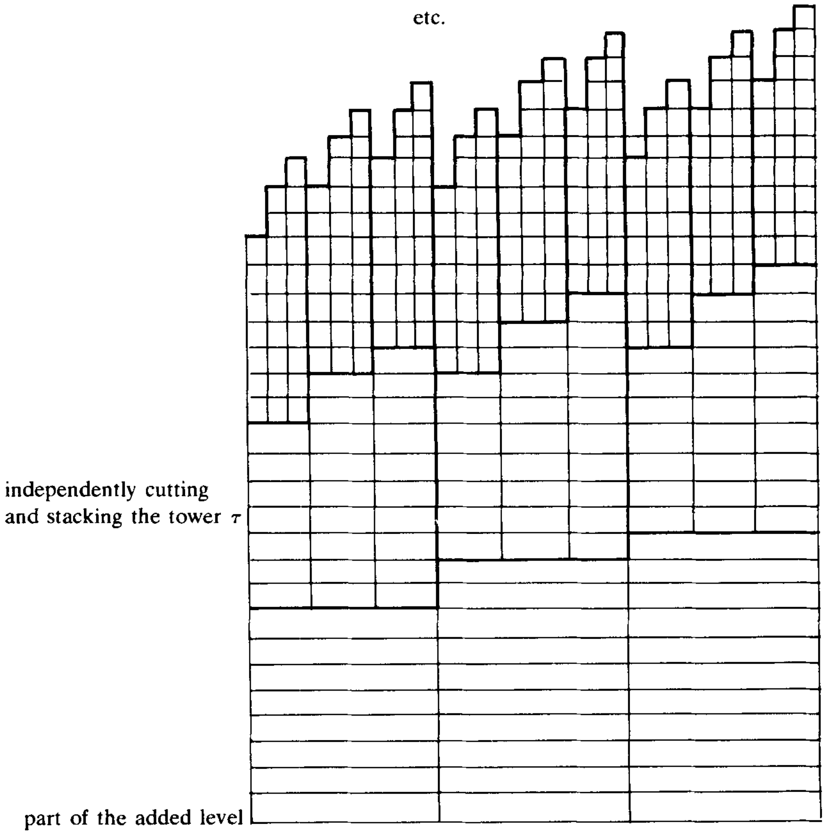
(c)





Stack the two α_{n+1}-subcolumns of successive columns in the n-tower in order, as indicated. Divide each half of the resulting column into $\alpha_{n+1}h'_{n+1}/r_n$ subcolumns of equal width. Then stack these as in Katok's example. The union of the $(1-2\alpha_{n+1})$-subcolumns of the columns in the n-tower is a tower isomorphic to the n-tower, $1-2\alpha_{n+1}$ the width of the n-tower. Call this tower τ. In order to do the independent stacking, τ will be divided into more towers isomorphic to the n-tower. (The picture has been expanded horizontally for diagrammatic purposes only.)

(d)



Step 1. Above the part of the added level where the transformation was not already defined in (c), place an isomorphic copy of the n-tower, $1/h'_{n+1}$ the width of the n-tower, constructed from τ.

Step 2. From what remains of τ after step 1, construct r_n towers isomorphic to the n-tower, each $1/r_n h'_{n+1}$ the width of the n-tower. Stack one of these towers above each of the columns of the tower constructed in step 1.

Step 3. From what remains of τ after step 2, construct r_n^2 towers isomorphic to the n-tower, each $1/r_n^2 h'_{n+1}$ the width of the n-tower. Stack one of these towers above each of the columns of the tower constructed in step 2.

Repeat for a total of $(1-2\alpha_{n+1})h'_{n+1}$ steps.

(The picture has now been drastically expanded in the horizontal direction, again for diagrammatic purposes only.)

Fig. 2. Making $(n+1)'$-tower from n-tower

call the 0-tower. (We call a union of a finite number of columns of intervals a tower, even if the columns have different heights.) We now describe how to get the $(n+1)'$-tower from the n-tower. Partition the n-tower into r_n columns according to the sequences of symbols obtained as we read labels of points in vertical lines. These columns will all have the same width. Divide each of these columns into three subcolumns with widths in proportion to α_{n+1}, α_{n+1}, $1-2\alpha_{n+1}$. Above the second subcolumn of the last column add another level and label it 0. The first subcolumn of each column is divided up further to construct the initial $\alpha_{n+1}h'_{n+1}$ n-blocks; the second subcolumn is similarly used to construct the second group of $\alpha_{n+1}h'_{n+1}$ n-blocks, with the 0 at the end of each cycle coming from the added level. The third subcolumn is used for the independent concatenation of n-blocks. (See Fig. 2.) The $(n+1)$-tower is obtained by breaking the $(n+1)'$-tower into some thinner towers isomorphic to itself, and stacking these in a manner similar to that shown in part (d) of Fig. 2.

For $n \geqq 0$, let F_n $[F'_{n+1}]$ be the set of points in the base of the n-tower $[(n+1)'$-tower$]$, let X_n be the set of points in the n-tower (which is the same as the set of points in the n'-tower, if $n \geqq 1$), and let ν_n be normalized Lebesgue measure on X_n. Let $\mu_n = \nu_n \times \nu_n$ on $X_n \times X_n$. Let $X = \bigcup_0^\infty X_n$, and let $\mathcal{P}$ be the partition of X into $h_0 + 1$ sets according to which of the labels $0, 1, \cdots, h_0$ a point has. Define a partition $\mathcal{R}_n$ $[\mathcal{R}'_n]$ of X_n by $\mathcal{R}_n$ $[\mathcal{R}'_n] = \{J: J$ is a level in a column of the n-tower $[n'$-tower$]$, i.e. J is the set of points in a certain position of a fixed n-block $[n'$-block$]\}$.

We now place some requirements on $h_0, h_1, h_2, \cdots$ to ensure that X will have finite Lebesgue measure. We will do this by forcing $\nu_n(X_n - X_{n-1})/\nu_n(X_{n-1})$ to be less than $\frac{1}{2}\delta_n$, for each $n \geqq 1$. (Note that $\sum_1^\infty \delta_n < \infty$.) For $n \geqq 1$, let l_n be the sum of the lengths of the $(n-1)$-blocks, i.e. the length of a cycle of $(n-1)$-blocks. Then

$$\frac{\nu_n(X_n - X_{n-1})}{\nu_n(X_{n-1})} = \frac{\alpha_n}{l_n}.$$

By a very crude estimate, we have $l_n > h_{n-1}$, for $n \geqq 2$; also, $l_1 = h_0$. Thus, by requiring $h_{n-1} > 2\alpha_n/\delta_n$, for $n \geqq 1$, we get

$$\frac{\nu_n(X_n - X_{n-1})}{\nu_n(X_{n-1})} < \tfrac{1}{2}\delta_n. \tag{1}$$

Hence, we can get X to have finite Lebesgue measure. Normalize this measure, call it ν, and let $\mu = \nu \times \nu$ on $X \times X$.

We now pause to obtain some related estimates involving μ_n, which we will need later. Note that

$$1-\mu_n(X_{n-1}\times X_{n-1})=1-\nu_n(X_{n-1})^2<2(1-\nu_n(X_{n-1}))<\frac{2(1-\nu_n(X_{n-1}))}{\nu_n(X_{n-1})}<\delta_n.$$

Then if $B\subset X_{n-1}\times X_{n-1}$,

$$\begin{aligned}\mu_n(B)&=\mu_n(X_{n-1}\times X_{n-1})\mu_{n-1}(B)\\ &\geqq \mu_{n-1}(B)-(1-\mu_n(X_{n-1}\times X_{n-1}))\\ &>\mu_{n-1}(B)-\delta_n.\end{aligned}\tag{2}$$

Define a partial automorphism T_n $[T'_{n+1}]$ on X_n $[X_{n+1}]$, $n\geqq 0$, by letting T_n $[T'_{n+1}]$ map each point in the n-tower $[(n+1)'$-tower$]$ to the point above it in the next level, if there is one; otherwise leave T_n $[T'_{n+1}]$ undefined there. Then $T_0\subset T'_1\subset T_1\subset T'_2\subset T_2\subset\cdots$. Let T be the transformation on X which is the common extension of T_0, T'_1, T_1, T'_2, $T_2,\cdots$. This defines T almost everywhere on X.

It can easily be seen from an inductive argument that for each n', if the $\mathcal{P}$-name of some $x\in X$ from j to k agrees with that of some n'-block then we must have $T^jx\in F'_n$ and $T^{k'}x\in F'_n$, where $k'=k+1$ if $T^{k+1}x\not\in\mathcal{P}_0$, $k'=k+2$ otherwise. (Note that 0's do not occur consecutively in any $\mathcal{P}$-name.) Hence, if $\mathcal{F}'_n$ is the partition $\{F'_n, X-F'_n\}$, then

$$\bigvee_j^k T^{-i}\mathcal{F}'_n\subset\bigvee_j^k T^{-i}\mathcal{P},$$

provided that $k-j$ is sufficiently large so that any $\mathcal{P}$-name of length $k-j+1$ must contain an entire n'-block. Thus $\mathcal{R}'_n\subset\bigvee_{-\infty}^{\infty}T^{-i}\mathcal{P}$ for each $n\geqq 1$. Since the $\mathcal{R}'_n$'s generate (the Lebesgue sigma-field), this implies that $\mathcal{P}$ generates under T. (Alternatively, we could simply restrict T to the sigma-field generated by $\mathcal{P}$.)

For almost every $x\in X$, there are infinitely many n such that $T^{-1}x$ lies in the initial $\alpha_n h'_n$ $(n-1)$-blocks in an n'-block. Now if x and n satisfy this condition, and if we are given the past times i at which $T^ix\in F'_n$, then the time zero $\mathcal{P}$-name of x is determined. But from the previous paragraph

$$\bigvee_{-\infty}^{-1} T^{-i}\mathcal{F}'_n\subset\bigvee_{-\infty}^{-1} T^{-i}\mathcal{P},\qquad\text{for each } n\geqq 1.$$

Hence, for almost every x, the past $\mathcal{P}$-name of x determines the time zero $\mathcal{P}$-name of x. Thus $h(T)=h(T,\mathcal{P})=0$. (This can also be seen via a name-counting argument.)

By Rothstein's arguments [11], T is a V transformation provided that $h_0, h'_1, h_1, h'_2, h_2,\cdots$ grow sufficiently rapidly. The idea of his argument is to approximate the $(T, \mathscr{P})$ process in finite-dimensional $\bar{d}$ by the Markov shifts obtained by concatenating n-blocks (or n'-blocks) independently. With the appropriate growth condition on the parameters (which depends on how quickly the VWB property of the Markov shifts constructed takes effect), the VWB property of the Markov shifts will copy over to yield property V for T.

§5. Proof that $T \times T$ is LB

This proof involves a refinement of the nesting techniques introduced in Weiss' Notes ([16], section 7) and in the work of Katok and Sataev [6]. Let $s_1 = 1$, $s_n = \min\,[1,\ 50\sqrt{\delta_{n-1}} + (1 - \alpha_n^4)s_{n-1}]$, $n = 2, 3, \cdots$. Note that $\lim_{n\to\infty} s_n = 0$, because $\Sigma_1^\infty \sqrt{\delta_n} < \infty$ and $\Sigma_1^\infty \alpha_n^4 = \infty$. We will show inductively that, for each $n \geqq 1$, there is a set $B_n \subset X_n \times X_n$ and an integer N_n such that

(3)
- (a) $\mu_n(B_n) > 1 - \delta_n$,
- (b) B_n is a union of atoms in $\mathscr{R}_n \times \mathscr{R}_n$,
- (c) each point of B_n is of the form (x, y), where x and y both have at least $N_n - 1$ points lying above them in the n-tower,
- (d) if $p, q \in B_n$, then $\bar{f}_{N_n}(p, q) \leqq s_n$.

(In this section we will always take the $\bar{f}$ distance with respect to $(T \times T)$-$(\mathscr{P} \times \mathscr{P})$-names.) More precisely, for each $n > 1$, if $h_0, h_1', h_1, h_2', h_2, \cdots, h_{n-1}', h_{n-1}$ have been chosen to make (3) true for $n - 1$, then by requiring h_n' and h_n to be sufficiently large, we can make (3) hold for n. At each stage, of course, h_n' and h_n may have to be made larger than what would be required to make (3) true, in order to obtain (1) and property V in addition.

We proceed with the proof of (3), which clearly implies that $T \times T$ is LB. For $n = 1$, (3) holds trivially by taking $B_1 = X_1 \times X_1$, $N_1 = 1$. Now assume $n > 1$, $h_0, h_1', h_1, h_2', h_2, \cdots, h_{n-1}', h_{n-1}$ have been chosen, and there exist B_{n-1}, N_{n-1} such that (3) holds with n replaced by $n - 1$. Choose L_n sufficiently large so that

$$\frac{l_n(l_n + 1)}{L_n} < \delta_n \quad \text{and} \quad \frac{N_{n-1}}{L_n} < \delta_n.$$

Require that h_n' be sufficiently large so that

$$\frac{L_n}{h_n'} < \delta_n.$$

Let A_n^1 be the set of x such that $x, Tx, \cdots, T^{L_n - 1}x$ are in the first group of $\alpha_n h_n'$ $(n-1)$-blocks in some n'-block, and let A_n^2 be the set of y such that y,

$Ty, \cdots, T^{L_n-1}y$ are in the part of some n'-block which consists of the second group of $\alpha_n h'_n$ $(n-1)$-blocks with a 0 added at the end of each cycle of $(n-1)$-blocks. Let $A_n = A_n^1 \times A_n^2$. Note that if $p = (x, y) \in A_n$, the T-$\mathcal{P}$-L_n-name of x has period l_n and the T-$\mathcal{P}$-L_n-name of y has period $l_n + 1$. Hence the $(T \times T)$-$(\mathcal{P} \times \mathcal{P})$-$L_n$-name of p has period $l_n(l_n + 1)$. It follows that if $p, q \in A_n$, then

$$\bar{f}_{L_n}(p, q) \leqq \frac{l_n(l_n+1)}{L_n} < \delta_n. \tag{4}$$

If K_n denotes the sum of the lengths of the first group of $\alpha_n h'_n$ $(n-1)$-blocks in an n'-block, then we have

$$\nu_{n-1}(A_n^1) = \alpha_n\left(1 - \frac{L_n - 1}{K_n}\right) > \alpha_n\left(1 - \frac{L_n}{\alpha_n h'_n}\right) > \alpha_n - \delta_n.$$

Similarly, $\nu_{n-1}(A_n^2 \cap X_{n-1}) > \alpha_n - \delta_n$. Then, using (2), we obtain

$$\mu_n(A_n) > \mu_{n-1}(A_n \cap (X_{n-1} \times X_{n-1})) - \delta_n > (\alpha_n - \delta_n)^2 - \delta_n > \alpha_n^2 - 3\delta_n. \tag{5}$$

Let $\mathcal{D}_n$ be the partition of F_n into r_n sets such that two points of F_n are in the same atom of $\mathcal{D}_n$ if and only if they are in the same column of the n-tower, i.e. they are in the same n-block. Let σ_n be a Bernoulli shift on F_n with independent generator $\mathcal{D}_n$. Define $\tilde{T}_n$ to be the extension of T_n to all of X_n obtained by mapping a point in the n-tower which has no point above it and which is in the same vertical line as $z \in F_n$ to $\sigma_n(z)$. Then $(\tilde{T}_n, \mathcal{R}'_n)$ is a Markov process corresponding to the independent stacking of n'-blocks. Note that once $h_0, h'_1, h_1, h'_2, h_2, \cdots, h'_{n-1}, h_{n-1}, h'_n$ are chosen, $(\tilde{T}_n, \mathcal{R}'_n)$ is isomorphic to the same Markov shift regardless of the choice of h_n. This Markov shift must be mixing because there are two n'-blocks whose lengths differ by one. We now apply the ergodic theorem twice to the product Markov process $(\tilde{T}_n \times \tilde{T}_n, \mathcal{R}'_n \times \mathcal{R}'_n)$. First note that A_n and B_{n-1} are both unions of atoms of $\mathcal{R}'_n \times \mathcal{R}'_n$. Also, by (2) and (3), we have

$$\mu_n(B_{n-1}) > \mu_{n-1}(B_{n-1}) - \delta_n > 1 - \delta_{n-1} - \delta_n > 1 - 2\delta_{n-1}.$$

The basic idea now is to choose h_n sufficiently large so that the occurrences of A_n and B_{n-1} are well-distributed in the $\mathcal{R}'_n \times \mathcal{R}'_n$-names of most points, in order to combine the $\bar{f}$-matchings obtained from A_n in (4) and from B_{n-1} in (3). Begin by choosing M_n sufficiently large so that $L_n / M_n < \delta_n$ and such that the set

$$\tilde{U}_n = \{(x,y) \in X_n \times X_n : \text{frequency of } (\tilde{T}_n \times \tilde{T}_n)^i(x,y) \text{ being in } A_n, \text{ for } 0 \leqq i \leqq M_n - 1, \text{ is greater than } \alpha_n^2 - 4\delta_n\} \tag{6}$$

has μ_n-measure greater than $1 - \delta_n$. Next choose N_n sufficiently large so that $M_n / N_n < \delta_n$ and such that the set

$$\begin{cases} \tilde{V}_n = \{(x,y) \in X_n \times X_n : \text{As } i \text{ ranges over } [0, N_n), \text{ frequency of } (\tilde{T}_n \times \tilde{T}_n)^i(x,y) \text{ being in } \tilde{U}_n \text{ is greater than } 1 - 2\delta_n, \text{ frequency of } (\tilde{T}_n \times \tilde{T}_n)^i(x,y) \text{ being in } A_n \text{ is greater than } \alpha_n^2 - 4\delta_n, \text{ and frequency of } (\tilde{T}_n \times \tilde{T}_n)^i(x,y) \text{ being in } B_{n-1} \text{ is greater than } 1 - 3\delta_{n-1}\} \end{cases} \tag{7}$$

has μ_n-measure greater than $1 - \frac{1}{2}\delta_n$. Now require h_n to be sufficiently large so that $N_n / h_n < \delta_n / 8$. (Note that even though we actually defined $\tilde{T}_n$ in terms of h_n, the choices of M_n, N_n depend only on the isomorphism class of $(\tilde{T}_n \times \tilde{T}_n, \mathcal{R}'_n \times \mathcal{R}'_n)$; so the order of choices is okay.) Let $G_{n,i}$ be the set of points in X_n having at least $i - 1$ points above them in the n-tower. If $(x,y) \in (G_{n,i} \times G_{n,i})$, then the $(\tilde{T}_n \times \tilde{T}_n)$-$(\mathcal{R}'_n \times \mathcal{R}'_n)$-$i$-names and the $(T \times T)$-$(\mathcal{R}'_n \times \mathcal{R}'_n)$-$i$-names of (x,y) agree. Let $U_n = \tilde{U}_n \cap (G_{n,M_n-1} \times G_{n,M_n-1})$. Then the points in U_n satisfy the frequency condition in (6) with $\tilde{T}_n$ replaced by T. Let

$$B_n = \tilde{V}_n \cap (G_{n,(N_n-1)+(M_n-1)} \times G_{n,(N_n-1)+(M_n-1)}).$$

Since $\nu_n(G_{n,(N_n-1)+(M_n-1)}) > 1 - (N_n + M_n)/h_n > 1 - 2N_n/h_n > 1 - \frac{1}{4}\delta_n$, we have

$$\mu_n(G_{n,(N_n-1)+(M_n-1)} \times G_{n,(N_n-1)+(M_n-1)}) > 1 - \tfrac{1}{2}\delta_n.$$

Then $\mu_n(B_n) > \mu_n(\tilde{V}_n) - \frac{1}{2}\delta_n > 1 - \delta_n$. If $(x,y) \in B_n$, the frequency conditions in (7) hold with $\tilde{T}_n$ replaced by T and $\tilde{U}_n$ replaced by U_n. Note that B_n is a union of atoms of $\mathcal{R}_n \times \mathcal{R}_n$, as required. Let $p, q \in B_n$. We will show that these frequency conditions imply that $\bar{f}_{N_n}(p,q) < s_n$.

The frequency of $i \in [0, N_n)$ such that $(T \times T)^i p \in A_n$ is greater than $\alpha_n^2 - 4\delta_n$. Partition the sequence of points $p, (T \times T)p, \cdots, (T \times T)^{N_n-1}p$ except for beginning and end into disjoint strings of length L_n. There are L_n ways of doing this, and for at least one of them, the frequency of L_n-strings which start with a point in A_n is greater than $\alpha_n^2 - 4\delta_n$. Fix such a choice of L_n-strings and let $S_1, S_2, \cdots, S_t$ be a listing, in order, of all of these strings except those which contain points within M_n of the end of the sequence $p, (T \times T)p, \cdots, (T \times T)^{N_n-1}p$. Since $(M_n + L_n)/(N_n - 2L_n) < 2\delta_n$, the frequency of $i \in [1, t]$ such that S_i starts with a point in A_n is greater than $\alpha_n^2 - 6\delta_n$. The frequency of $i \in [0, N_n)$ such that $(T \times T)^i q \in U_n$ is greater than $1 - 2\delta_n$. Write down the sequence q, $(T \times T)q, \cdots, (T \times T)^{N_n - 1}q$ directly below the sequence p, $(T \times T)p, \cdots$, $(T \times T)^{N_n-1}p$. Since all but at most $2L_n + M_n$ of the sequence p, $(T \times T)p, \cdots, (T \times T)^{N_n-1}p$ is contained in some S_i, $i \in [1, t]$, and $(2L_n + M_n)/N_n < 2\delta_n$, the frequency of $i \in [1, t]$ such that some point written below S_i is in U_n is greater than $1 - 4\delta_n$. Thus, for $i \in [1, t]$, with frequency greater than $\alpha_n^2 - 10\delta_n$ we have both that S_i starts with a point in A_n and that there is a point written below S_i that is in U_n. Fix, for the moment, an i such that these two conditions hold. Let $S_i = ((T \times T)^a p, \cdots, (T \times T)^{a+L_n-1}p)$ and let $j \in [0, L_n)$ be such that $(T \times T)^{a+j}q \in U_n$. Then, by the definition of U_n, the frequency of $m \in [0, M_n)$ such that $(T \times T)^{a+j+m}q \in A_n$ is greater than $\alpha_n^2 - 4\delta_n$. Since $j/M_n < \delta_n$, this implies that the frequency of $m \in [0, M_n)$ such that $(T \times T)^{a+m}q \in A_n$ is greater than $\alpha_n^2 - 5\delta_n$. We now shift the sequence q, $(T \times T)q, \cdots, (T \times T)^{N_n-1}q$ to the left by $M_n - 1$ units, one unit at a time, and apply a "Fubini argument" as in Weiss' Notes ([16], fig. 1, p. 7.13). From this we see that there exists $m \in [0, M_n)$ such that if we shift the sequence $q, (T \times T)q, \cdots, (T \times T)^{N_n-1}q$ to the left by m units, then for $i \in [1, t]$, with frequency greater than $(\alpha_n^2 - 10\delta_n)(\alpha_n^2 - 5\delta_n)$, both S_i and the string directly below S_i start with a point in A_n. Let I_1 be the set of $i \in [1, t]$ for which this is true. For the rest of the argument, we leave the bottom string shifted m units to the left relative to the top string. For each $i \in [1, t]$, let the string below S_i be $\bar{S}_i$. By (4), for $i \in I_1$,

$$\bar{f}_{L_n}(S_i, \bar{S}_i) < \delta_n. \tag{8}$$

Let

$$\gamma = \frac{|I_1|}{t} > (\alpha_n^2 - 10\delta_n)(\alpha_n^2 - 5\delta_n) > \alpha_n^4 - 15\delta_n. \tag{9}$$

Since all but at most $2\delta_n$ of the points in the sequence p, $(T \times T)p, \cdots, (T \times T)^{N_n-1}p$ are contained in some S_i, and the frequency of being in B_{n-1} among points in this sequence is greater than $1 - 3\delta_{n-1}$, we know that among the j's such that $(T \times T)^j p$ is in some S_i, the frequency of being in B_{n-1} is greater than $1 - 5\delta_{n-1}$. Because this is also true if p is replaced by q and S_i by $\bar{S}_i$, it follows that among the j's such that $(T \times T)^j p$ is in some S_i, the frequency that both $(T \times T)^j p$ and the point below $(T \times T)^j p$, namely $(T \times T)^{j+m}q$, are in B_{n-1} is greater than $1 - 10\delta_{n-1}$. Hence, for $i \in [1, t]$, with frequency greater than $1 - \sqrt{10\delta_{n-1}}$, S_i satisfies the condition that with frequency greater than $1 - \sqrt{10\delta_{n-1}}$, a point in S_i and the point in $\bar{S}_i$ below it are both in B_{n-1}. Let I_2 be the set of $i \in [1, t] - I_1$ for which this is true. Then

$$\frac{|I_2|}{t-|I_1|} > 1 - \frac{\sqrt{10\delta_{n-1}}}{1-\gamma}. \tag{10}$$

We now fix $i \in I_2$ and estimate $\bar{f}(S_i, \bar{S}_i)$. Partition all of S_i except beginning and end into strings of length N_{n-1} such that with frequency greater than $1 - \sqrt{10\delta_{n-1}}$, a string starts with a point such that both it and the point listed below it are in B_{n-1}. Let these strings be $S_i^1, \cdots, S_i^r$, and let the strings directly below them be $\bar{S}_i^1, \cdots, \bar{S}_i^r$. Then $\bar{f}_{N_{n-1}}(S_i^j, \bar{S}_i^j) \leqq s_{n-1}$, $j \in [1, r]$, with frequency greater than $1 - \sqrt{10\delta_{n-1}}$. Since $N_{n-1}/L_n < \delta_n$, all but at most $2\delta_n$ of the points in S_i are in some S_i^j, $j \in [1, r]$, and we have that for $i \in I_2$,

$$\bar{f}_{L_n}(S_i, \bar{S}_i) < s_{n-1} + \sqrt{10\delta_{n-1}} + 2\delta_n < s_{n-1} + 6\sqrt{\delta_{n-1}}. \tag{11}$$

Combining (8), (9), (10), and (11), we finally obtain

$$\begin{aligned}\bar{f}_{N_n}(p,q) &< 2\delta_n + \gamma\delta_n + [1-\gamma]\left[\frac{\sqrt{10\delta_{n-1}}}{1-\gamma} + s_{n-1} + 6\sqrt{\delta_{n-1}}\right] \\ &< 13\sqrt{\delta_{n-1}} + (1-\gamma)s_{n-1} \\ &< 13\sqrt{\delta_{n-1}} + (1-[\alpha_n^4 - 15\delta_n])s_{n-1} \\ &< 28\sqrt{\delta_{n-1}} + (1-\alpha_n^4)s_{n-1}.\end{aligned}$$

Thus, $\bar{f}_{N_n}(p,q) \leqq s_n$. □

§6. Modifying the construction of T to get all n-fold Cartesian products $T \times \overset{n}{\cdots} \times T$ to be LB

Let $\alpha_1, \alpha_2, \cdots, \delta_1, \delta_2, \cdots$ be as before, except we now require that $\sum_{n=1}^{\infty} \alpha_n^j = \infty$ for *each* positive integer j. We inductively define a sequence $s_1, s_2, \cdots$ in $(0,1]$ and a sequence of nonnegative integers $k_0, k_1, \cdots$. Let $k_0 = 0$. For each positive integer j, let

$$s_{k_{j-1}+1} = 1; \quad s_n = \min[1, 50\sqrt{\delta_{n-1}} + (1-\alpha_n^{2j})s_{n-1}], \qquad n = k_{j-1}+2, \cdots, k_j$$

where k_j is chosen so that $k_j \geqq k_{j-1}+2$, $s_{k_j} < 1/j$, and for $n > k_j$, $(j+1)\alpha_n < 1/(j+1)$. For $n \geqq 1$, define j_n to be the integer such that $k_{j_n-1} < n \leqq k_{j_n}$. Then we have $j_n\alpha_n < 1/j_n$; hence $\lim_{n\to\infty} j_n\alpha_n = 0$. This will be used to ensure that T satisfies property V. The construction of T will be described in terms of $\alpha_1, \alpha_2, \cdots, \delta_1, \delta_2, \cdots, j_1, j_2, \cdots$, and the positive integer parameters $h_0, h'_1, h_1, h'_2, h_2, \cdots$. To make certain that the space on which T will be defined has finite measure, we need the following.

LEMMA. *Given $\varepsilon > 0$ and a positive integer j, there exists a positive integer $M = M(\varepsilon, j)$ such that if N is an integer with $N \geqq M$, then there exist positive integers $m_1, \cdots, m_j$ such that $N + m_1$, $N + m_2, \cdots, N + m_j$ are pairwise relatively prime and $m_i/N < \varepsilon$, $1 \leqq i \leqq j$.*

The proof of this lemma is easy and will be omitted.

We proceed with the "block construction" for T. The 0-blocks are defined as before, where we require $h_0 > \max(2, \alpha_1/\delta_1)$, and we again construct $(n+1)'$-blocks from n-blocks, and $(n+1)$-blocks from $(n+1)'$-blocks, for $n \geqq 0$. Again let l_{n+1} be the sum of the lengths of the n-blocks; let r_n $[r'_n]$ be the number of n-blocks [n'-blocks].

We now indicate how to build $(n+1)'$-blocks from n-blocks, for $n \geqq 0$. Choose positive integers $m_{n+1,1}, \cdots, m_{n+1,j_{n+1}}$ such that $l_{n+1} + m_{n+1,1}, \cdots, l_{n+1} + m_{n+1,j_{n+1}}$ are pairwise relatively prime and

$$\frac{m_{n+1,i}}{l_{n+1}} < \frac{\delta_{n+1}}{j^2_{n+1}\alpha_{n+1}}, \qquad \text{for } 1 \leqq i \leqq j_{n+1}.$$

(It will be shown inductively that this is always possible; see the next paragraph. In the case $n = 0$, this is clear, since $j_1 = 1$ and $l_1 = h_0$, and we can simply take $m_{1,1} = 1$.) We require that h'_{n+1} be chosen so that $\alpha_{n+1}h'_{n+1}$ is a multiple of r_n. To build the $(n+1)'$-blocks, first order the n-blocks in some manner and let $\alpha_{n+1}h'_{n+1}$ n-blocks go in cyclically with $m_{n+1,1}$ 0's added at the end of each cycle; $\cdots$ let $\alpha_{n+1}h'_{n+1}$ n-blocks go in cyclically with $m_{n+1,j_{n+1}}$ 0's added at the end of each cycle; finally, let $(1 - j_{n+1}\alpha_{n+1})h'_{n+1}$ n-blocks go in independently.

As before, we construct $(n+1)$-blocks by concatenating h_{n+1} $(n+1)'$-blocks independently. We require that $h_{n+1} > M(\delta_{n+2}/j^2_{n+2}\alpha_{n+2}, j_{n+2})$, where the function M is defined as in the lemma above. Since $l_{n+2} > h_{n+1}$, this condition will enable us to now construct $(n+2)'$-blocks from $(n+1)$-blocks so that (12) is satisfied with $n+2$ replacing $n+1$.

The cutting and stacking process associated to this block construction is similar to the previous one. However, to obtain the $(n+1)'$-tower from the n-tower, each of the r_n columns of the n-tower is now divided into $j_{n+1}+1$ subcolumns with widths in proportion to $\alpha_{n+1}, \cdots, \alpha_{n+1}$, $1 - j_{n+1}\alpha_{n+1}$. Above the

ith subcolumn of the last column, $1 \leqq i \leqq j_{n+1}$, $m_{n+1,i}$ levels are added, and the points in these levels are labeled 0. The ith subcolumn of each column, $1 \leqq i \leqq j_{n+1}$, is further divided up to construct the ith group of $\alpha_{n+1} h'_{n+1}$ n-blocks, with the $m_{n+1,i}$ 0's at the end of each cycle coming from the added levels. The $(j_{n+1}+1)$th subcolumn of each column is used for the independent concatenation of n-blocks.

Define F_n, F'_n, X_n, X, ν_n, ν, T_n, T'_n, T, $\mathscr{P}$, $\mathscr{R}_n$, $\mathscr{R}'_n$ as before. Let $\mu^j_n = \nu_n \times \overset{j}{\cdots} \times \nu_n$, and $\mu^j = \nu \times \overset{j}{\cdots} \times \nu$. Note that by (12) we have

$$\frac{\nu_n(X_n - X_{n-1})}{\nu_n(X_{n-1})} < (j_n \alpha_n)(\delta_n / j_n^2 \alpha_n) = \delta_n / j_n. \tag{13}$$

In particular, this implies that X has finite Lebesgue measure. Also, if $B \subset X_{n-1} \times \overset{j_n}{\cdots} \times X_{n-1}$ then

$$\begin{aligned}
\mu^{j_n}_n(B) &\geqq \mu^{j_n}_{n-1}(B) - (1 - \mu^{j_n}_n(X_{n-1} \times \overset{j_n}{\cdots} \times X_{n-1})) \\
&= \mu^{j_n}_{n-1}(B) - (1 - \nu_n(X_{n-1})^{j_n}) \\
&> \mu^{j_n}_{n-1}(B) - j_n(1 - \nu_n(X_{n-1})) \\
&> \mu^{j_n}_{n-1}(B) - \delta_n.
\end{aligned} \tag{14}$$

The same arguments as before show that T has entropy zero and that T is a V transformation, provided $h_0, h'_1, h_1, h'_2, h_2, \cdots$ grow sufficiently rapidly.

To prove that $T \times \overset{j}{\cdots} \times T$ is LB for each positive integer j, we will establish the following variant of (3). For each $n \geqq 1$, there is a set $B_n \subset X_n \times \overset{j_n}{\cdots} \times X_n$ and an integer N_n such that

(15)
- (a) $\mu^{j_n}_n(B_n) > 1 - \delta_n$,
- (b) B_n is a union of atoms of $\mathscr{R}_n \times \overset{j_n}{\cdots} \times \mathscr{R}_n$,
- (c) each point of B_n is of the form $(x_1, \cdots, x_{j_n})$ where each x_i, $1 \leqq i \leqq j_n$, has at least $N_n - 1$ points lying above it in the n-tower,
- (d) if $p, q \in B_n$, then $\bar{f}^{j_n}_{N_n}(p, q) \leqq s_n$.

Here $\bar{f}^j$ denotes the $\bar{f}$ distance with respect to $(T \times \overset{j}{\cdots} \times T)$-$(\mathscr{P} \times \overset{j}{\cdots} \times \mathscr{P})$-names.

Fix, for the moment, a positive integer j. We will indicate why (15) implies that $T \times \overset{j}{\cdots} \times T$ is LB. Choose $n = k_m$, where $m \geqq j$. Then $j_n = m$, and we have a set $B_n \subset X_n \times \overset{m}{\cdots} \times X_n$ with $\mu_n^m(B_n) > 1 - \delta_n$ such that $\bar{f}_{N_n}^m(p, q) \leqq s_n$ if $p, q \in B_n$. Let $\pi: X_n \times \overset{m}{\cdots} \times X_n \to X_n \times \overset{j}{\cdots} \times X_n$ be the projection onto the first j coordinates. Then $\mu_n^j(\pi(B_n)) > 1 - \delta_n$, and if $p, q \in \pi(B_n)$, $\bar{f}_{N_n}^j(p, q) \leqq s_n$.

We proceed with the proof of (15). If $n = k_{j_n - 1} + 1$, then $s_n = 1$ and (15) is trivially satisfied. Assume that $k_{j_n-1} + 1 < n \leqq k_{j_n}$ and that there exist B_{n-1}, N_{n-1} such that (15) holds with n replaced by $n - 1$. At this point $h_0, h_1', h_1, h_2', h_2, \cdots, h_{n-1}', h_{n-1}$ have been chosen, and we must now choose h_n', h_n so that (15) is satisfied.

Choose L_n sufficiently large so that

$$\frac{(l_n + m_{n,1}) \cdots (l_n + m_{n,j_n})}{L_n} < \delta_n \quad \text{and} \quad \frac{N_{n-1}}{L_n} < \delta_n.$$

Require that h_n' be sufficiently large so that

$$\frac{L_n}{h_n'} < \varepsilon_n, \tag{16}$$

where ε_n is chosen so that $(\alpha_n - \varepsilon_n)^{j_n} > \alpha_n^{j_n} - 2\delta_n$. For $1 \leqq i \leqq j_n$, let A_n^i be the set of $x \in X_n$ such that $x, Tx, \cdots, T^{L_n - 1}x$ are in the ith group of $\alpha_n h_n'$ $(n-1)$-blocks in some n'-block. Let $A_n = A_n^1 \times \cdots \times A_n^{j_n}$. Suppose $p = (x_1, \cdots, x_{j_n}) \in A_n$. Then the T-$\mathcal{P}$-L_n-name of x_i has period $l_n + m_{n,i}$, $1 \leqq i \leqq j_n$; hence the $(T + \overset{j_n}{\cdots} \times T)$-$(\mathcal{P} \times \overset{j_n}{\cdots} \times \mathcal{P})$-$L_n$-name of x has period $(l_n + m_{n,1}) \cdots (l_n + m_{n,j_n})$. Thus, if $p, q \in A_n$, then

$$\bar{f}_{L_n}^{j_n}(p, q) \leqq \frac{(l_n + m_{n,1}) \cdots (l_n + m_{n,j_n})}{L_n} < \delta_n. \tag{17}$$

Also, we have $\nu_{n-1}(A_n^i \cap X_{n-1}) > \alpha_n - L_n / h_n' > \alpha_n - \varepsilon_n$. Then, using (14), we obtain

$$\mu_n^{j_n}(A_n) > \mu_{n-1}^{j_n}(A_n \cap X_{n-1} \times \overset{j_n}{\cdots} \times X_{n-1}) - \delta_n > (\alpha_n - \varepsilon_n)^{j_n} - \delta_n > \alpha_n^{j_n} - 3\delta_n, \tag{18}$$

which is analogous to (5).

Fortunately, the previous nesting argument still works here, if we make the following minor changes. We change all Cartesian products to j_n-fold Cartesian products, change each α_n^2 to $\alpha_n^{j_n}$, and require that h_n be sufficiently large so that

$N_n/h_n < \varepsilon'_n$, where ε'_n is chosen so that $(1-\varepsilon'_n)^{j_n} < 1 - \frac{1}{2}\delta_n$. This completes the proof of (15). □

§7. Generalization to flows

We now obtain an example of a flow $\phi = \{\phi_t : t \in \mathbf{R}\}$ which is V and has $\phi \times \cdot \overset{n}{\cdot} \cdot \times \phi = \{\phi_t \times \cdot \overset{n}{\cdot} \cdot \times \phi_t : t \in \mathbf{R}\}$ LB for each positive integer n. This will be done by slightly altering the preceding construction and building a flow over the resulting transformation T and under a function g.

Let $\alpha_1, \alpha_2, \cdots, \delta_1, \delta_2, \cdots, s_1, s_2, \cdots, k_0, k_1, \cdots, j_1, j_2, \cdots$, be as described in section 6. The present construction will be described in terms of these parameters and the positive integer parameters $h_0, h'_1, h_1, h'_2, h_2, \cdots$ (which may be different from those in section 6).

For each $n \geqq 0$, we will construct n-towers [$(n+1)'$-towers] for T and for ϕ, which will be referred to as T-n-towers [T-$(n+1)'$-towers] and ϕ-n-towers [ϕ-$(n+1)'$-towers], respectively. Let X_n be the set of points in the T-n-tower (or, equivalently, if $n \geqq 1$, the T-n'-tower, since no material will be added in constructing the T-n-tower from the T-n'-tower). We will obtain the ϕ-n'-tower from the T-n'-tower as follows. Suppose $C, TC, \cdots, T^sC$ are the levels constituting a column in the T-n'-tower. The function g will be defined on X_n so that g is constant on each T^iC. Let this constant be $g(T^iC)$. Then stack the rectangles $C \times [0, g(C)), \cdots, T^sC \times [0, g(T^sC))$ in order. We will take this to be the corresponding column of the ϕ-n'-tower. The construction of the ϕ-n-tower from the T-n-tower is similar. For $n \geqq 1$, let l_n^g denote the sum of the heights of the columns in the ϕ-$(n-1)$-tower.

We define a T-0-block to consist of a single symbol chosen from $\{1, \cdots, h_0\}$, $h_0 > \max(1, \alpha_1/\delta_1)$, and we build the corresponding T-0-tower. Let β be an irrational number greater than one. Define g on the T-0-tower by $g = \beta$ at those points corresponding to the symbol 1, and $g = 1$ elsewhere. Then the ϕ-0-tower is obtained in the manner described above.

Assume $n \geqq 0$ is fixed for the moment and that the T-n-tower and the ϕ-n-tower have been constructed. Also assume that g has been defined on X_n so that g is constant on each level of each column in the n-tower. If $n \geqq 1$, assume in addition that numbers

$$m_{1,1}, \cdots, m_{1,j_1}, m_{2,1}, \cdots, m_{2,j_2}, \cdots, m_{n,1}, \cdots, m_{n,j_n}$$

have been chosen so that

$$(19)\quad \begin{cases} \text{(i) } 1, \beta, m_{1,1}, \cdots, m_{1,j_1}, m_{2,1}, \cdots, n_{2,j_2}, \cdots, m_{n,1}, \cdots, m_{n,j_n} \text{ are linearly independent over the rationals,} \\ \text{(ii) for each } i \text{ such that } 1 \leqq i \leqq n, \\ \qquad \dfrac{1}{l_i^g + m_{i,1}}, \cdots, \dfrac{1}{l_i^g + m_{i,j_i}} \\ \text{are linearly independent over the rationals,} \\ \text{(iii) } m_{i,t+1} - m_{i,t} > 1 \text{ if } 1 \leqq i \leqq n \text{ and } 1 \leqq t < j_i, \text{ and} \\ m_{i+1,1} - m_{i,j_i} > 1 \text{ if } 1 \leqq i < n, \\ \text{(iv) } m_{i,j_i} < j_1 + \cdots + j_i \text{ if } 1 \leqq i \leqq n. \end{cases}$$

The construction of T-$(n+1)'$-blocks from T-n-blocks proceeds as before, except that only one 0 is added after each cycle of n-blocks in the beginning of the T-$(n+1)'$-blocks. Choose $m_{n+1,1}, \cdots, m_{n+1,j_{n+1}}$ so that (19) is still true with $n+1$ replacing n. The T-$(n+1)$-blocks are again constructed by independently concatenating h_{n+1} T-$(n+1)'$-blocks. Here we require

$$h_{n+1} > \frac{(j_1 + \cdots + j_{n+1}) j_{n+1}^2 \alpha_{n+1}}{\delta_{n+1}}.$$

Build the T-$(n+1)'$-tower and the T-$(n+1)$-tower corresponding to these block constructions. Now define g on $X_{n+1} - X_n$ by setting g equal to $m_{n+1,i}$ at those points in the T-$(n+1)'$-tower corresponding to 0's added after each cycle in the ith group of $\alpha_{n+1} h'_{n+1}$ n-blocks in the beginning of a T-$(n+1)'$-block. Then g is constant on each level of each column in the T-$(n+1)'$-tower.

Let $X = \bigcup_0^\infty X_n$, and for each $A \subset X$, define $A^g = \{(x,t): x \in A, 0 \leqq t < g(x)\}$. We will identify X_n^g with the points in the ϕ-n-tower (or, equivalently, the ϕ-n'-tower if $n \geqq 1$). Let ϕ be the flow on X^g built over T. Then ϕ is the common extension of the partially defined flows ${}_n\phi$ $[{}_{n'}\phi]$ obtained by flowing upwards at unit speed in the ϕ-n-tower [ϕ-n'-tower].

For each $n \geqq 0$, let ν_n be the normalized Lebesgue product measure on X_n^g. Then

$$(20)\qquad \frac{\nu_n(X_n^g - X_{n-1}^g)}{\nu_n(X_{n-1}^g)} < \frac{\max\limits_{(1 \leqq i \leqq j_n m_{n,j})} j_n \alpha_n}{l_n^g} < \frac{(j_1 + \cdots + j_n) j_n \alpha_n}{h_{n-1}} < \frac{\delta_n}{j_n}.$$

Hence X^g has finite Lebesgue product measure; normalize it and call it ν. (Since g is bounded away from zero, X must therefore have finite Lebesgue measure.) Define μ_n^j, μ^j as before. By the same computation as for (14) we see that if

$B_n \subset X_n^g \times \overset{j_n}{\cdots} \times X_n^g$, then $\mu_n^{j_n}(B) > \mu_{n-1}^{j_n}(B) - \delta_n$. Let $\mathcal{P}$ be the partition of X according to which of the labels $\{0, 1, \cdots, h_0\}$ a point has, and let $\mathcal{P}^g = \{P^g : P \in \mathcal{P}\}$ be the corresponding partition of X^g. For each n $[n']$, let the ϕ-n-blocks [ϕ-n'-blocks] be the continuous ϕ-$\mathcal{P}^g$-names of points in the bases of the columns of the ϕ-n-tower [ϕ-n'-tower], with the block lengths taken equal to the heights of the corresponding columns. Let $\mathcal{R}_n$ $[\mathcal{R}'_n]$ be the sigma-field on X_n^g generated by sets of the form $\{(x, t): x \in C, t_1 \leqq t \leqq t_2\}$, where C is the base of a column in the ϕ-n-tower [ϕ-n'-tower] and $0 \leqq t_1 < t_2 < g(C)$.

By the same argument as before, $\mathcal{P}$ generates under T and $h(T) = h(T, \mathcal{P}) = 0$. Thus $h(\phi) = 0$.

We now indicate why ϕ is V.

It can be seen inductively that for each $n \geqq 0$, there exists a number c which is a linear combination of $1, \beta, m_{1,1}, \cdots, m_{1,j_1}, \cdots, m_{n,1}, \cdots, m_{n,j_n}$ over the nonnegative integers such that one ϕ-n-block has length $c + 1$ and another has length $c + \beta$. (This is also true for the ϕ-n'-blocks, if $n \geqq 1$.) From (19(i)), it follows that $c + 1$ and $c + \beta$ are linearly independent over the rationals. In particular, this implies that the lengths of ϕ-n-blocks [ϕ-n'-blocks] are not arithmetic (i.e., do not lie in an arithmetic progression containing zero). Let ${}_n\tilde{\phi}$ $[{}_{n'}\tilde{\phi}]$ be the flow on the ϕ-n-tower [ϕ-n'-tower] built over σ_n $[\sigma_{n'}]$, a Bernoulli shift on the base with independent generator consisting of the partition of the base according to the columns of the tower. (Then ${}_n\tilde{\phi}$ $[{}_{n'}\tilde{\phi}]$ is an extension of ${}_n\phi$ $[{}_{n'}\phi]$.) The flows ${}_n\tilde{\phi}$ and ${}_n\tilde{\phi}$ are Bernoulli flows (see Ornstein [8], page 60). For the rest of this paper, we fix a choice of $\gamma > 0$ which is small compared to $m_{1,1}$, the minimum value of g. Then if h_0, h'_1, h_1, h'_2, $h_2, \cdots$ grow sufficiently rapidly, property V can be obtained by approximating $(\phi_\gamma, \mathcal{P}^g)$ by $({}_n\phi_\gamma, \mathcal{P}^g)$ and $({}_{n'}\phi_\gamma, \mathcal{P}^g)$ in finite $\bar{d}$. We will show next that $\mathcal{P}^g$ generates (the product Lebesgue sigma-field) under ϕ_γ. Thus ϕ_γ is V and, consequently, ϕ is V.

Note that the 0's in the T-$\mathcal{P}$-name of a point $x \in X$ cannot occur consecutively, and for each other symbol, there is only one corresponding value of g. This implies that for each $(x, t) \in X^g$, the continuous ϕ-$\mathcal{P}^g$-name of (x, t) determines both the T-$\mathcal{P}$-name of x and the value of t. Hence the continuous ϕ-$\mathcal{P}^g$-names separate points in X^g. Since the different values g assumes on points of X labeled 0 are bounded apart by 1, and γ is small compared to the minimum value of g, the discrete ϕ_γ-$\mathcal{P}^g$-name of any point in X^g determines the continuous ϕ-$\mathcal{P}^g$-name up to a possible translate of at most γ. Thus, to show that the ϕ_γ-$\mathcal{P}^g$-names separate points in X^g, it suffices to show that for each $(x, t) \in X^g$, the times at which the continuous ϕ-$\mathcal{P}^g$-name of (x, t) changes from one symbol to another are dense mod γ (i.e., in $\mathbf{R}/\gamma\mathbf{Z}$). Since the lengths of the

ϕ-n-blocks are not arithmetic, some ϕ-n-block has length η_n, where η_n is not a rational multiple of γ. Choose k_n such that $\eta_n, 2\eta_n, \cdots, k_n\eta_n$ are δ_n-dense mod γ. Now if $h'_{n+1} \geqq k_n/(1-j_{n+1}\alpha_{n+1})$, there is some ϕ-$(n+1)'$-block in which the above ϕ-n-block of length η_n occurs consecutively k_n times. Thus, if this ϕ-$(n+1)'$-block occurs in the ϕ-$\mathcal{P}^g$-name of a point (x,t), then the times at which the ϕ-$\mathcal{P}^g$-name changes are δ_n-dense mod γ. But this happens for every $(x,t) \in X^g$, because, by construction, each ϕ-$(n+2)'$-block contains each ϕ-$(n+1)'$-block. Since $\lim_{n\to\infty}\delta_n = 0$, this implies that if $h_0, h'_1, h_1, h'_2, h_2, \cdots$ grow sufficiently rapidly, then the times at which the continuous ϕ-$\mathcal{P}^g$-name of any point in X^g changes are dense mod γ. Consequently, $\mathcal{P}^g$ generates under ϕ_γ.

We now turn to the proof that $\phi \times \cdot \overset{j}{\cdot} \cdot \times \phi$ is LB for each positive integer j. A flow is defined to be LB if some (and hence every) cross-section is LB. It is shown in the Weiss Notes [16] that this is equivalent to each of the following:

(i) each transformation in the flow is LB (or has LB components),

(ii) some transformation in the flow is LB.

Thus it suffices to show that $(\phi_\gamma \times \cdot \overset{j}{\cdot} \cdot \times \phi_\gamma, \mathcal{P}^g \times \cdot \overset{j}{\cdot} \cdot \times \mathcal{P}^g)$ is LB for each j. We do this by showing that (15) holds with X_n replaced by X_n^g, the $\bar{f}^j$ distance now denoting the $\bar{f}$ distance with respect to $(\phi_\gamma \times \cdot \overset{j}{\cdot} \cdot \times \phi_\gamma)$-$(\mathcal{P}^g \times \cdot \overset{j}{\cdot} \cdot \times \mathcal{P}^g)$-names, and with (c) replaced by

(c′) each point of B_n is of the form $((x_1,t_1), \cdots, (x_{j_n}, t_{j_n}))$ where $t_i + \gamma(N_n - 1)$ is less than the length of the ϕ-n-block containing (x_i, t_i), $1 \leqq i \leqq j_n$.

This argument is very similar to that in section 6, so we will only briefly indicate the necessary modifications. The case $n = k_{j_n-1}+1$ is again trivial; so assume $k_{j_n-1}+1 < n \leqq k_{j_n}$ and there exist B_{n-1}, N_{n-1}, so that the analogue of (15) given above holds with n replaced by $n-1$. To choose L_n, proceed as follows. Let $\xi_{n,i} = l_n^g + m_{n,i}$ and let ${}_n\psi$ be the flow at unit speed on each coordinate of $\mathbf{T}_{\xi_{n,1}} \times \cdots \times \mathbf{T}_{\xi_{n,j_n}}$, where $\mathbf{T}_\xi = \mathbf{R}/\xi\mathbf{Z}$, with normalized Lebesgue measure. Put a partition $\mathcal{Q}_{n,i}$ on $\mathbf{T}_{\xi_{n,i}}$ such that the continuous $\mathcal{Q}_{n,i}$-${}_n\psi$-name of length $\xi_{n,i}$ of $0 \in \mathbf{T}_{\xi_{n,i}}$ is obtained by listing the ϕ-$(n-1)$-blocks in order once (the order being determined by the corresponding T-$(n-1)$-blocks) and then adding an interval of length $m_{n,i}$ labeled 0; i.e., the $\mathcal{Q}_{n,i}$-${}_n\psi$-name of length $\xi_{n,i}$ is the same as one period in the ith type of cycle in the beginning of a ϕ-n'-block. Since ${}_n\psi_\gamma$ is isomorphic to a rotation by $(\gamma/\xi_{n,1}, \cdots, \gamma/\xi_{n,j_n})$ on $\mathbf{T} \times \cdots \times \mathbf{T}$, and by (19 (ii)) $\gamma/\xi_{n,1}, \cdots, \gamma/\xi_{n,j_n}$ are linearly independent over the rationals, we have, if L_n is sufficiently large, $\bar{f}^{j_n}_{L_n, {}_n\psi_\gamma}(x,y) < \delta_n$ for all $x, y \in \mathbf{T}_{\xi_{n,1}} \times \cdots \times \mathbf{T}_{\xi_{n,j_n}}$. (Here we are taking the $\bar{f}$-distance between ${}_n\psi_\gamma$-$(\mathcal{Q}_{n,1} \times \cdots \times \mathcal{Q}_{n,j_n})$-names of length L_n.) Require, in addition, that L_n be sufficiently large so that $N_{n-1}/L_n < \delta_n$. Instead of (16), require that h'_n be sufficiently large so that

$$L_n\gamma/h'_n < \varepsilon_n$$

where ε_n is again chosen so that $(\alpha_n - \varepsilon_n)^{j_n} > \alpha_n^{j_n} - 2\delta_n$. For $1 \leqq i \leqq j_n$, let A_n^i be the set of $x \in X_n$ such that $\{\phi_t(x)\colon 0 \leqq t \leqq (L_n - 1)\gamma\}$ is contained in the ith type of cycle in the beginning of a ϕ-n'-block. Let $A_n = A_n^1 \times \cdots \times A_n^{j_n}$. Then $A_n \in \mathcal{R}'_n \times \cdots \times \mathcal{R}'_n$ and we again have

$$\bar{f}^{j_n}_{L_n,\phi_\gamma}(p,q) < \delta_n \quad \text{if } p,q \in A_n \quad \text{and} \quad \mu_n^{j_n}(A_n) > \alpha_n^{j_n} - 3\delta_n.$$

Consider again the Bernoulli flow ${}_n\tilde{\phi}$ defined earlier, and, in particular, ${}_n\tilde{\phi}_\gamma$. The ergodic theorem is applied twice to the process $({}_n\tilde{\phi}_\gamma \times \overset{j_n}{\cdots} \times {}_n\tilde{\phi}_\gamma, \mathcal{A}_n \vee \mathcal{B}_{n-1})$, where $\mathcal{A}_n = \{A_n, X_n - A_n\}$, $\mathcal{B}_{n-1} = \{B_{n-1}, X - B_{n-1}\}$, in order to set up the nesting argument as in section 5. Note that since $\mathcal{A}_n \vee \mathcal{B}_{n-1}$ is $\mathcal{R}'_n \times \overset{j_n}{\cdots} \times \mathcal{R}'_n$-measurable, the distribution of the $({}_n\tilde{\phi}_\gamma \times \overset{j_n}{\cdots} \times {}_n\tilde{\phi}_\gamma, \mathcal{A}_n \vee \mathcal{B}_{n-1})$ process does not depend on h_n. We then select h_n, and finish the argument as in section 6, with a few obvious changes. □

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